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ABSTRACT

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Formal Operations and the Ability of Students to Structure Geometric Content

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Paper presented at the Annual Meeting of the Jean Piaget Society

Philadelphia, Pennsylvania, June 2, 1983

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Abstract

The ability of tenth grade plane geometry students to structure concepts and relationships from a geometry unit on ratio, proportion and similarity was tested. Analysis of cognitive "maps" of the structural relationships possessed by twenty concrete and twenty formal subjects indicated that formal operational subjects structure subject matter content significantly more like subject matter experts and teachers than do concrete operational subjects. Results also showed that the cognitive maps of students more closely approximate those of subject matter experts than those of their own teachers.

Introduction

Learning Geometry

Despite persistent attempts to improve the content and instruction of geometry, Kinsella (1965) noted that failures in tenth grade geometry were greater than in any other high school subject. Consistent disenchantment of students with geometry, as evidenced by a decrease in enrollments, and an increase in student failures and teacher frustrations instigated many curricular changes in geometry between the years 1900 and 1950 (NCTM, 1970a).

A consistent complaint of geometry students is the amount of memorization involved in the learning of geometry. Yet, although a certain amount of memorization is involved, the extent of memorization should not be as great as is indicated by students (NCTM, 1930, 1970a). Rather, national committees of mathematics educators have recommended shifts in emphasis from the previous memorization of "book theorems" to increased work with original exercises and proofs. Increased emphasis on numerical and practical applications has also been recommended. Yet students and teachers appear to continue to rely on memorization as a primary strategy for the learning and teaching of geometry (NCTM, 1970a, 1970b).

This apparent reliance on rote memorization could explain the resulting dislike of geometry and poor performance of students. It is possible that there are qualities inherent in the content of high school geometry which make it difficult, or impossible, for certain students to

develop meaningful cognitive structures of its concepts. As a result, these students may be forced to learn the material by rote. It is possible that such students have not yet reached an appropriate level of cognitive development needed to create these essential cognitive structures. The hypothesized level of development necessary for meaningful structuring and learning of this geometric content is Piaget's formal operational stage.

Substantiation of this view is found in Renner: "During [the tenth grade], the student normally makes his first contact with Euclidean geometry. Finding a better example of a secondary school course that requires more formal operational thinking than does Euclidean geometry is difficult" (Renner et al., 1976, pp.96-97). Since significant portions of students in high school mathematics are not fully formal (Collis, 1975; Farmer et al., 1982; Renner et al., 1976; Shayer & Adey, 1981), the implications for mathematics educators are significant.

Forming Cognitive Structures

When a child first encounters concepts in geometry, the child learns terms such as triangle, square and circle. Although, mathematically speaking, these concepts do not "exist" in a real world sense, there are many concrete models available which can be used to symbolize or model these mathematical concepts. The child eventually learns to differentiate between the concepts, based upon their relative attributes (right triangles versus isosceles triangles), and to recognize class inclusion principles (all rectangles are parallelograms). As the student learns more geometry, s/he is required to deal with increasing levels of

abstraction. Relational operators such as equal, congruent and similar gain prominence. The student is forced to learn not just concepts but the rules involving the interrelationships between those concepts (the corresponding sides of similar triangles are proportional).

Thus, students are continually restructuring the concepts, causing the internalized "web," or network of concepts to undergo reorganization. Factors which impact upon the restructuring of concepts include both the subject matter itself and the characteristics of the learner. Several investigators have noted the degree to which students restructure representations of concepts during an instructional sequence (Johnson, 1969; Shavelson, 1974; Thro, 1978). Other researchers (Stasz, Shavelson, Cox & Moore, 1976) have studied the relationship between such individual differences variables as field independence and restructuring. However, the relationship between representation and formal operations has received comparatively little attention (Preece, 1978). The purpose of this research was to investigate more fully that relationship.

Theoretical Bases

Developmental Perspective

From a cognitive, developmental perspective, the term "structure" plays a significant role in the nomological net. Development proceeds through a series of qualitatively different stages in which each stage is characterized by a different type of cognitive structure. These

structures determine what an individual can or cannot do at any particular stage of development. Structures develop successively with later structures building upon and incorporating the structures of earlier stages. This view of cognitive structure is essentially Piagetian. Knowledge is acquired and built up over time through physical and mental interactions with objects, people, and events.

There are, however, essential differences between the Piagetian concept of structure and those of others. Perhaps the major difference is that for Piaget, cognitive structures, or schemata, are generalizable capabilities which are neither content nor domain specific. They represent a conceptual framework through which all knowledge is acquired or constructed. The cognitive construction is based upon the simultaneous interaction of assimilation and accommodation. Information is taken in and made to fit the structures already present in the individual; and, at the same time, through the process of accommodation, the child creates new structures. Through this constant interaction of assimilation and accommodation, cognitive growth occurs. The knowledge which is acquired through these processes is united with the processes through which it was acquired to form an organization (Piaget, 1952, 1954).

Piaget employs mathematical models to describe the developing schemata in the child. These logico-mathematical structures are conceived of as models of cognitive structures. The cognitive schemata available to the formal operational thinker represent a complex type of cognitive structure and represent a large improvement over the

capabilities of the concrete operational thinker. The new structures available allow, for the first time, hypothetico-deductive thinking. Also, because of the newly acquired ability for combinatorial analysis, the formal thinker is able to generate and consider all possible combinations of variables. This ability to relate terms in a combinatorial manner is joined by a new ability to reason about the possible and not just the actual. As a result of the ability to deal with the real as a subset of the possible, the formal thinker develops an ability to perform operations on operations. The availability of hypothetical reasoning, combinatorial analysis and propositional thought further allows the individual to dissociate variables and deal with proportions (Piaget, 1972). It is these cognitive schemata which are hypothesized as being necessary for the structuring of the concepts and relationships of geometry. Piaget states that subject matter must be taught in "forms assimilable to children of different ages in accordance with their mental structure and the various stages of development" (1970, p. 153).

Neimark notes that "although Piaget's treatment of formal operations provides a promising theoretical framework, its utility...is hampered by [a] lack of coordination of theoretical constructs to experimental operations" (1975, p. 586). Descriptions of cognitive structures detailed by Ausubel (1964) and Bruner (1965) were guilty of similar operational shortcomings. From the late 1960s until the present, however, several real advances have been made in attempts to operationalize further these theoretical constructs and to develop

measurement instruments believed to assess both the structure of content and the cognitive structures possessed by learners. These more recent investigations of cognitive structure are described below.

Representations of Cognitive Structure

The term cognitive structure has been used as a hypothetical construct referring to the organization of concepts in an individual's long-term memory. It is the individual's idiosyncratic, internalized representation of a given content following learning. The measurement of the structure is based on the assumptions which are made regarding how the concepts are stored in memory.

Some of the research in the methodologies used in this study originated in the late 1950s and early 1960s with linguistic theorists including James Deese (1962, 1969). Further methodological developments were made by Garskof and Houston (1963) and P.E. Johnson (1965, 1967, 1969).

Although the majority of the work in this field has involved the subject matters of physics and mechanics, some work, notably that of Geeslin and Branca, has involved mathematics. To date, some intriguing results have come from this body of research. Work by Shavelson and Geeslin (1975) showed that achievement in and learning of physics and mathematics content result in changes in cognitive structure and can be assessed in word-association tests and similarity judgments. They found through visual analysis that the clustering of concepts from a content scaling was very similar to cognitive structures obtained following

instruction. Further work (Geeslin & Shavelson, 1975) showed that subjects also retained these structures over a period of time. P.E. Johnson (1967) found significant differences between the structures of low and high achievers in physics. Geeslin (1974) found a strong similarity between the representation of content structure and the structure of the subjects following instruction in mathematical probability.

Stasz et al., (1976) have done research which indicates a high degree of alignment between the cognitive structures of students and subject matter experts. Hambleton and Sheehan (1977) have noted measureable differences in students' cognitive structures as a result of instruction. Others have found a high degree of correspondence among the structures of curriculum developers and content structures (Shavelson & Geeslin, 1975). Several studies have also shown that student structures change following instruction, and that these changes bring the student structures closer to those of content experts (Branca, 1975; Glazer-Waldeman & Cox, 1980; Johnson, 1965; Thro, 1978). Branca (1980) examined the communication of mathematical structure from curriculum developers to teachers and then to students. By visually inspecting student structures resulting from Johnson's hierarchical clustering analysis, he concluded that there was a high degree of correspondence between the cognitive structure of teachers, the curriculum developers and the content. He also found close matches between the cognitive structure of a class and that of its particular teacher, and clear distinctions between the structures of low and high achievers.

Representing Structure : Multidimensional Scaling

Multidimensional scaling has been utilized as a technique for indicating how individuals structure domains of knowledge. In "mapping" cognitive structure, the aim is to obtain structural information about the internal representation of concepts. It is assumed that experts in a field can generate a complex and accurate set of inter-relationships which can be used as a model of the subject-matter structure. Although geometry can be characterized as a semantically rich domain (Simon, 1979) because of the high number of possible relationships among its concepts, experts typically agree when they relate concepts. It is, therefore, possible to develop "expert" structures of a subject matter content and use them as a basis of comparison with corresponding student structures of the same content.

Methodology

Subjects

Subjects consisted of 161 students from six tenth grade Regents geometry classes from a suburban New York State school. Approximately one-half of the students were in one of three geometry classes taught by one teacher (Teacher A). The rest of the subjects were in one of three geometry classes taught by a second teacher (Teacher B). All of the subjects studying with Teacher A used the text, Geometry, by Hirsch et al., (1979). All of the students of Teacher B used the text, Geometry by Jurgensen et al., (1975). These two texts are parallel in their axiomatic development of the theorems within the similarity unit.

Instruments

Measures of Formal Operations Sheehan's (1970) version of the Longeot test (1964) and the Test of Logical Thinking (TOLT) (Tobin & Capie, 1981) were utilized to assess cognitive stage. The Longeot test was particularly appropriate because of its emphasis on the logic of deductions. The TOLT was chosen because it measures five separate formal abilities. The research design required the determination of groups of "concrete" and "formal" subjects. Here, the terms were used in a global context rather than being used to identify specific, separate formal operational abilities. Therefore, an effort was made to include as many separate abilities as possible in the testing. As a result, group membership in the "formal" or "concrete" groups was based on a mean standardized score combined from the Longeot and TOLT instruments.

Measures of Cognitive Structure

Two separate instruments were designed to map expert and student structure. The first instrument developed (Task A) was a similarity judgment task built from thirteen terms chosen from the similarity unit in geometry. For this task, the subject was shown two terms at a time and was asked to make a numerical judgment of the relationship between the terms. The numbers ranged from 1 to 10 where "1" represented very little or no relationship; "4" a limited relationship; "7" a strong relationship and "10" that they were identical.

Subjects were given examples using terms from other geometry units to aid them in their decisions. They were then given all possible pairs of

terms from the following: ratio, proportion, equal, similar, congruent, equilateral triangle, right triangle, parallel, corresponding sides, corresponding angles, isosceles triangle, geometric mean, and polygon. The terms for the comparisons were chosen from the key terms in the two texts (Jurgensen et al., 1975, Hirsch et al., 1979). An original list of sixteen terms was decreased to thirteen following field testing and interviews with five "experienced" geometry teachers. Expert opinion was sought with regard to the significance, degree of interrelatedness, and consistency of the judgment about the terms. Five teachers comprised the expert group and together devised consensus judgments which were used to create the expert targets.

The unit of ratio, proportion, and similarity was chosen as the unit of study for several reasons. Since the unit occurs approximately half-way through the course, this ensured that non-trivial proof would be well established in the course. The student would have, at this point, a reasonably large pool of definitions, postulates, and theorems with which to deal. The theorems of the unit are also derived substantially from the definitions, theorems, and postulates within the similarity unit itself.

Most importantly, the course and unit were chosen for the large number of correspondences between their concepts and the Piagetian model of formal operational structure. Perhaps the most significant reason for choosing the unit was its emphasis on the concept of proportionality. The ability to recognize and apply the laws of proportions is a major characteristic of the formal operational thinker (Inhelder & Piaget,

1958, p.314). Also, Piaget, Inhelder and Szeminska (1960) suggest that the idea of proportions be sought through the perception of similar figures, making this unit an even more "natural" choice. Proportionality is also an important element in the other formal schemata and Piaget shows logically how proportion can be used to interconnect all of the formal INRC group (Inhelder & Piaget, 1958, pp. 65-66, pp. 178-181).

The second task (Task B) was designed in an attempt to "ferret out" the structural relationships in the subject's memory beyond the syntactic level. With this instrument, subjects were given pairs of mathematical symbols to compare and were asked to make relational judgments using the same scale as for the first task. The items which the subjects were asked to compare in this task were mathematical expressions and relationships like AB:DE=AC:DF and $\measuredangle A \cong \measuredangle D$. Represented within the total set of expressions were most of the same terms and concepts from the first list, but at a symbolic (and hopefully, more abstract) level (Bloom, 1956). It was assumed that in order for the subject to choose an appropriate number, the subject had to be familiar with the mathematical symbols, theorems, postulates, and definitions which corresponded to the given expression and to the mathematical, structural relationship between the expressions.

For example, the subject was given symbols representing two sets of triangles which were similar, but only some of the given proportions were directly related to that particular set of triangles. It was assumed that the subject would have to go beyond the pure syntactical relationship of proportion and similar triangles to give an appropriate response.

Subjects were given all combinations of pairs from ten expressions. The expressions were chosen in the same fashion as the terms for the first task. The list was decreased to 10, based on field testing and the judged complexity of the task by subject matter experts. The 45 comparisons given to the subjects were based on all combinations of the following expressions: $\triangle ABC \sim \triangle DEF$, $\triangle ABC \sim \triangle EDF$, $\triangle ABC \cong \triangle DEF$, $\angle A \cong \angle D$, $\overline{AB} \parallel \overline{DE}$, $AB/DE = AC/DF$, $AC/BA = DF/ED$, $AB/AC = DE/EF$, $(AB)(DF) = (DE)(AC)$, and $(AB+AC)/AC = (DE+EF)/EF$.

Reliability

Each of the two tasks involving similarity judgments were administered to 23 subjects on two separate occasions, 18 days apart, in a school district and class comparable to those in the major study. The students' similarity judgments were converted to distance matrices and subjected to multidimensional scaling analysis using MDSCALE (Rohlf et al., 1977). The resulting cognitive maps from the first administration were correlated with the structures obtained from the second administration using MXCOMP (Rohlf et al., 1977). The rank order correlations obtained were then converted to z-scores using Fischer's r to z transformation. The resulting z-scores were correlated, obtaining reliability coefficients of .705 and .773 for Tasks A and B, respectively.

Procedure

Subjects were administered the Longeot test of formal operations and the Test of Logical Thinking (TOLT) in December of the school year to

determine their level of cognitive development. Raw scores on the Longeot and TOLT were determined, standardized, and averaged to form a mean z-score. On the basis of this standard score, 40 subjects were chosen for the purpose of testing the hypotheses involving formal versus concrete subjects. The top 20 standard scores comprised the formal group and the lowest 20 formed the concrete group. The choice of 20 was determined by the fact that only the lowest 20 standard scores were classified as concrete operational on both measures of cognitive development, although the top 43 standard scores were classified as formal operational on both measures. The choice of using the standard score as well as the two classifications imposed the desired restrictions on the choice of the two classification groups representing polar extremes.

Following an instructional unit on geometric similarity, subjects were given the two instruments designed to map cognitive structure of the geometric content. This unit took place during the final weeks of January.

Prior to the testing of subjects on the four instruments, each of the instructors (Teachers A and B) was administered the two structuring tasks. The teachers were given the same instructions as the students and were asked to complete the instruments in a similar setting and time frame as the students. The structuring patterns of the instructors as well as the consensus matrices of subject matter experts were used as targets for subsequent analysis and comparison.

Hypotheses

It was hypothesized that the mean correlation between the expert consensus matrix and the matrices of formal operational subjects would be significantly greater than the mean correlations between the expert consensus matrix and the matrices of concrete operational subjects. Likewise, it was hypothesized that the mean correlations between the instructor matrices and formal operational subjects would be significantly greater than the mean correlation between the instructor matrices and the matrices of concrete operational subjects.

Previous research by Branca (1980) and Glazer-Waldeman and Cox (1980) also suggested an hypothesis that the correlations between mean subject matrices and the matrix of their instructor would be significantly higher than correlations with the other instructor or with the expert consensus matrix.

Results

Data Analysis

Multidimensional scaling analyses were performed on the similarity judgments of the experts and each teacher to determine the number of dimensions through which to perform the student analyses. Representations were developed for the teachers and expert consensus with 2,3, and 5 dimensions. A scree test (Catell, 1966) was performed on the resulting stress indicators to determine the analysis which would yield the best fit and would maximize the potential for interpretability. A

four dimensional target was chosen for Task A, which yielded an average stress of .0712 for the 40 subjects. Although a one- dimensional model would have been sufficient for the expert consensus on Task B, the stress factors for the two teachers indicated a choice of at least three dimensions.

For the subjects who were classified as formal or concrete, individual analysis of their structures was performed. First, the similarity matrices from each of the first two tasks were analyzed for each subject. The similarity matrices were standardized and distance coefficients were determined for each concept pair. The Euclidean distance algorithm was the a priori assumption (Shepard, 1980). The resulting distance matrices were subjected to nonmetric multidimensional scaling analysis (Kruskal, 1964) using MDSCALE (Rohlf et al., 1977). This analysis yields a measure of stress (or goodness of fit) of the distances in the configuration space to the monotone function of the original distances. Using the instructor matrices as a target, correlations were then determined between the instructor matrix and each of the formal and concrete matrices. Group matrices of mean scores were also determined for students of Teachers A and B and for the formal and concrete subjects. These group matrices were also correlated with each target.

Since the chosen number of dimensions did not represent the best fit model for all subjects, additional analyses were performed at lower and higher dimensional levels for subjects whose stress levels indicated a poor fit. The resulting optimal solutions, however, did not, in general,

increase the resemblence of the students to the targets.

Results

Comparisons with the Expert Targets

The rank order correlations between each of the students and the three targets were determined, converted to z-scores, averaged and tested for significance using a one tailed t-statistic.

Insert Table 1 About Here

The results for Task A are summarized in Table 1. As seen in the table, the maps of the formal subjects correlated significantly higher with each of the three targets (Experts, and Teachers A and B) than the maps of the concrete subjects. Each of these differences was significant with p .001. Note also, that the means for both the concrete and formal subjects are higher with the expert target than with their own teachers. This pattern also exists in the data for Task B (Table 2) and will be discussed in greater detail later.

The data for Task B are summarized in Table 2. As with Task A, formal subjects' matrices correlated significantly higher than concrete subjects' matrices against each of the three targets.

Insert Table 2 About Here

Prototypical Matrices

As was noted above, the mean correlations for Task A and B (Tables 1 and 2, respectively) were determined by taking the mean of the individual student correlations with each target. Another approach to the analysis was used to determine the mean similarity judgments for each group and then correlate that matrix of mean distances with each of the targets. Although the mean distances are sensitive to extreme scores in the data, this approach also has several advantages. With the first approach, although an attempt was made to find a number of dimensions which would provide the best fit for the most individuals, some variability in the correlations would be due to the number of dimensions chosen. If instead, one considers the matrix of mean similarities as being representative of a prototypical subject from that group, then it is possible to obtain rank order correlations between that single matrix and each of the targets. The advantage here is that it is more probable that a good dimensional model can be found for that individual matrix. It is also possible then to summarize group tendencies by scaling those mean distances.

For the data, prototypical matrices were determined and correlated with each of the targets for the concrete and formal subjects. The resulting correlations were consistent with the previous results. That is, the correlations with each of the three targets were higher for the prototypical formal subject than for the prototypical concrete subject.

These data are summarized in Table 3 (for Task A) and Table 4 (for Task B).

Insert Tables 3 and 4 About Here

It is worth noting that the correlations with the experts for both the concrete and formal prototype matrices increased substantially over the individual results. This is most probably due to the fact that both are derived from mean distances and both are, in that respect, prototypical.

Comparisons with Teacher Targets

The hypothesis that student judgments would approximate their own instructor's more closely than the other instructor's or the expert's was not confirmed. Rather, a different, quite distinct pattern emerged in the data. For all three tasks, although many patterns existed, the mean scores of all subjects were closest to the expert target and were furthest away from whichever teacher was furthest from the experts. Interestingly, this occurred irrespective of the teacher. For example, for Task A, Teacher A was closer to the expert target than Teacher B ($\underline{r}$ = .582 versus .536). And the mean scores for each of the groups followed the pattern:

$$\underline{r} \text{ w/Experts} > \underline{r} \text{ w/Teacher A} > \underline{r} \text{ w/Teacher B.}$$

For Task B subjects, in general, were closest to the experts and next closest to Teacher B. But, for both of these tasks, Teacher B was closest to the expert target. For these two tasks the following pattern emerged:

$$\underline{r} \text{ w/Experts} > \underline{r} \text{ w/Teacher B} > \underline{r} \text{ w/Teacher A.}$$

In general, then, the subjects were closer to the experts than to either teacher, and they were next closest to whichever teacher was closest to the experts. This pattern emerged for concrete and formal subjects (Tables 1 through 4) as well as for students of both Teachers A and B (Tables 5 and 6).

Insert Tables 5 and 6 about here

Discussion

Concrete Versus Formal Subjects

The significant differences in the cognitive representations of the concrete subjects and the expert target can be seen by comparing Figures 1 through 4. Although the scaling analyses of Tasks A and B were performed in four and three dimensions, respectively, the representations

are shown in two dimensions for ease of interpretability and discussion

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Insert Figures 1 and 2 about here

Task A In the expert map for Task A (Figure 1), the vertical dimension appears to represent a range of equality, where the positive pole represents most equal and the negative pole represents least equal quantities. Note that the term "corresponding sides" is much closer to the term "proportion," while the term "corresponding angles" is much closer to the terms "equal" and "congruent." This is consistent with the fact that "corresponding sides of similar triangles are proportional," while the "corresponding angles are congruent." The polygons are also placed consistently within this pattern indicating the trichotomy relationship between equilateral, isosceles and right triangles. Note the proximity of the term "right triangle" with the term "proportion," corresponding to the many similarity theorems involving proportions which are direct results of right triangle theorems. The horizontal axis appears to represent a whole versus part dimension. The "whole" figures, such as polygons and triangles, are at the positive extreme, and their "parts" (sides, angles) are at the negative extreme.

The cognitive representations of the teachers and students strongly indicate similar dimensional meaning. For the concrete subject, however, the whole versus part dimension is much more confused and compressed

(Figure 2). Note that the terms "parallel" and "equilateral triangle" are reversed on that axis. The concrete subject also places the terms "corresponding sides" and "corresponding angles" in close proximity. This is, perhaps, because of the syntactical proximity of the two terms, despite the fact that their roles are much more distinct in the similarity unit. In fact, close analysis of the entire upper left quadrant of the concrete map indicates a general confusion of several significant terms. It appears as though the concrete subjects are identifying some of these terms with their use in chapters involving congruent, rather than similar figures.

The prototypical formal representation of the geometry concepts (not shown) more closely approximated the expert target. Although the formal subjects made a clearer distinction than the concrete subjects between corresponding sides and angles they are still not as distinct as in the expert target. The major difference between the two representations was the location of the term "parallel." In fact, the term "parallel" was the term whose position varied the greatest from map to map. This may have been because this term is only marginally related to most of the terms in the unit.

Insert Figures 3 and 4 about here

Task B The expert representation of Task B (Figure 3) was actually one-dimensional. At the positive end of the vertical axis, one finds all

relationships involving triangles ABC and EDF while the relationships involving Δ ABC and Δ DEF are at the negative pole. However, the addition of extra dimensions was necessary for a good fit with the two teachers and most students.

Although the intent of designing the second task was to approach the content at a "higher" level, the results did not indicate that this was accomplished. Many concrete subjects approximated the map of the experts quite well. In fact, several even had correlations with the experts which exceeded those of the teachers with the experts. This result may be due to the fact that even though the task involves abstract mathematical symbolism, it is a routine instructional procedure to instruct students in an algorithm for dealing with the symbols. For example, students are shown that if Δ ABC $\sim$ Δ EDF, that they can form an appropriate proportion by placing the first two letters (AB) over the first and third letters (AC) and setting them equal to the first two letters of the second triangle (ED) over the first and third letters of that same triangle (EF). Thus, it is quite possible that concrete subjects were capable of matching appropriate symbolism by using a purely rote procedure. Since a similar algorithmic procedure was not available to them for Task A, this might account for the differential amount of success for concrete subjects on the two tasks.

An analysis of the statement pairs of Task B showed that concrete subjects were furthest from the target for concept pairs which included the statement Δ ABC $\cong$ Δ DEF. This is consistent with the difficulty which concrete subjects seemed to have in making a clear distinction

between congruence and similarity and congruence and proportionality in Task A. It seems reasonable that this difficulty would exist, especially since so little instructional time is spent on the similarity notions in comparison with that devoted to the congruence concepts. Additional concept pairs which had large variance and differed greatly from the mean were concept pairs dealing with proportional statements. This was particularly true when one proportion resulted from the first pair of triangles and the other proportion resulted from the second pair of triangles. For these concept pairs subjects often judged them as highly related, when they were not mathematically so.

Teacher Targets

It was hypothesized that students would approximate the structure of their own teacher more closely than that of the other teacher or the experts. As noted earlier, however, students' representations most closely approximated the expert consensus target, and second most closely approximated the teacher who was closer to the experts. This pattern was consistent, whether or not it was their own teacher. The formation of this hypothesis was based primarily on previous research by Glazer-Waldeman and Cox (1980) and Branca (1980). Since these data appear to contradict their results, a closer analysis of their data is warranted.

Branca's study of 109 secondary school mathematics students investigated their structuring of concepts of mathematical operational systems. This study indicated "a high degree of correspondence...between

the cognitive structure of a high school class and that of its particular teacher" (p.46). Glazer-Waldeman found similar results. However, both authors relied heavily on "visual interpretation" of the data for their conclusions (p.46). In fact, Branca makes a point of emphasizing the need to develop a meaningful statistical index to indicate the degree to which structures are similar (p.47). It is possible, then, that in these authors' interocular approaches, they were unable to detect significant differences in teacher and student structures.

Two other differences in the approaches of the studies might also account for the different results here. Branca used Johnson's hierarchical clustering procedure, which may not be as sensitive to differences. Glazer-Waldeman utilized the same MDSCALE procedure as this study, but failed to make use of the statistical comparison, MXCOMP. As such, their judgments of closeness of structure was a purely visual one. Another very important difference between their studies and this one is that they did not use a consensus of experts as a target. Rather, they compared classes to individual teachers and those classes "appeared" to match their own instructors best.

Another explanation is also tenable. Since the expert target was formed by a consensual process, it is likely that it represents a prototypical model of the subject matter content. This average schema represents none of the individual experts' cognitive representations but, rather, a composite of five similar ones. As such, the expert consensus does not contain the individual idiosyncrasies of any given teacher, as would the targets of Teachers A or B. Since individuals also develop

such idiosyncratic models as they learn concepts, it is logical to assume that individual structures would more closely approximate a mean, prototypical consensus structure. The student structure would represent a composite of relationships learned from teachers, texts, and peer interactions. And, although this student representation would have its own idiosyncratic features, it, in general, would more closely resemble a prototype than a second idiosyncratic model (Hayes-Roth, 1977; Thorndyke & Hayes-Roth, 1979). Since the Branca (1980) study was based on a seven-page text and a one-hour lecture, it is plausible that students were not exposed to the content to a degree sufficient to develop prototypical schemata. Given a longer period of time and a more complex set of concepts, their representations may have been less like that of their individual instructor.

Instructional Implications

The instructional implications of the research results and the Piagetian model are significant. First of all, they imply the necessity for screening of students early in the school year for cognitive level as a diagnostic procedure. Although one might be tempted to suggest the delay of instruction for concrete subjects, it is precisely the interaction with abstraction which should help promote the cognitive development desired. Rather than delay of instruction, the theory would imply a change in instructional modes and strategies to match the cognitive development of the student. Instruction of the concrete operational subject must include adequate concrete instances of

abstract concepts as well as concrete applications of rules.

For the similarity unit, this might suggest that the teacher integrate student manipulation of materials designed to assist in the attainment of the concepts of similarity and proportionality. The concrete subjects in the study had not yet clearly distinguished the concepts of congruence and similarity, nor those of equality and proportionality. Real examples involving shadows, projections, and indirect measurements might aid in these distinctions. Student construction of proportional lengths using compass and straight-edge, or even ruled edges, might help to promote this understanding.

In addition to changes in instructional implementation, the teacher might use the theoretical model to determine the cognitive complexity of the subject matter and attempt to impact curriculum (Farrell & Farmer, 1980; Shayer & Adey, 1981). Careful analysis of the content would suggest which prerequisite knowledge and experiences are necessary for the student to be able to begin to assimilate the new concepts into their cognitive structure.

The data would also suggest the necessity of testing students beyond the levels of knowledge and comprehension and of including levels of application, synthesis, analysis, and evaluation (Bloom, 1956). Otherwise, teachers will continue to encourage and promote memorization as long as it appears to be an effective strategy.

The implications of the results related to long-term retention and transfer are significant. Research has shown that the availability of cognitive schema increases the accessibility of a concept in memory and

aids in its retention (Thorndyke and Hayes-Roth, 1979). Other research has shown, quite convincingly, the long-term effect of rote memorization. To be learned, retained, and transferred, a meaningful structure of relevant concepts and principles must be built. This research suggests that, unless sufficient attention is placed on the development of that structure for the concrete student, teachers will continue to encourage students to memorize geometry (or any other abstract mathematical content) and just as quickly forget it.

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Table 1

Task A Mean Correlations of Distance Matrices

Comparing Subjects with Targets

TARGET	CONCRETE n=20	FORMAL n=20	PROB.
Experts	.211	.517	$p < .001$
Teacher A	.130	.349	$p < .001$
Teacher B	.075	.206	$p < .001$

Table 2

Task B Mean Correlations of Distance Matrices

Comparing Subjects with Targets

TARGET	CONCRETE n=20	FORMAL n=20	PROB.
Experts	.377	.770	$p = .011$
Teacher A	.177	.349	$p = .014$
Teacher B	.211	.459	$p = .004$

Table 3

Group Analyses for Task A - Prototypical Representations
Correlations of Mean Distance Matrices

TARGET	CONCRETE n=20	FORMAL n=20
Experts	.348	.605
Teacher A	.186	.262
Teacher B	.132	.075

Experts vs Teacher A r = .582
Experts vs Teacher B r = .536

Table 4

Group Analyses for Task B - Prototypical Representations
Correlations of Mean Distance Matrices

TARGET	CONCRETE n=20	FORMAL n=20
Experts	.806	.962
Teacher A	.398	.485
Teacher B	.578	.608

Experts vs Teacher A r = .427
Experts vs Teacher B r = .546

Table 5

Group Analyses of Task A by Teacher
Correlations of Mean Distance Matrices

TARGET	Students of Teacher A n=19	Students of Teacher B n=21
Experts	.577	.462
Teacher A	.231	.233
Teacher B	.192	.092

Experts vs Teacher A r = .582
Experts vs Teacher B r = .536

Table 6

Group Analyses of Task B by Teacher
Correlations of Mean Distance Matrices

TARGET	Students of Teacher A n=19	Students of Teacher B n=21
Experts	.419	.961
Teacher A	.547	.467
Teacher B	.559	.619

Experts vs Teacher A r = .427
Experts vs Teacher B r = .546

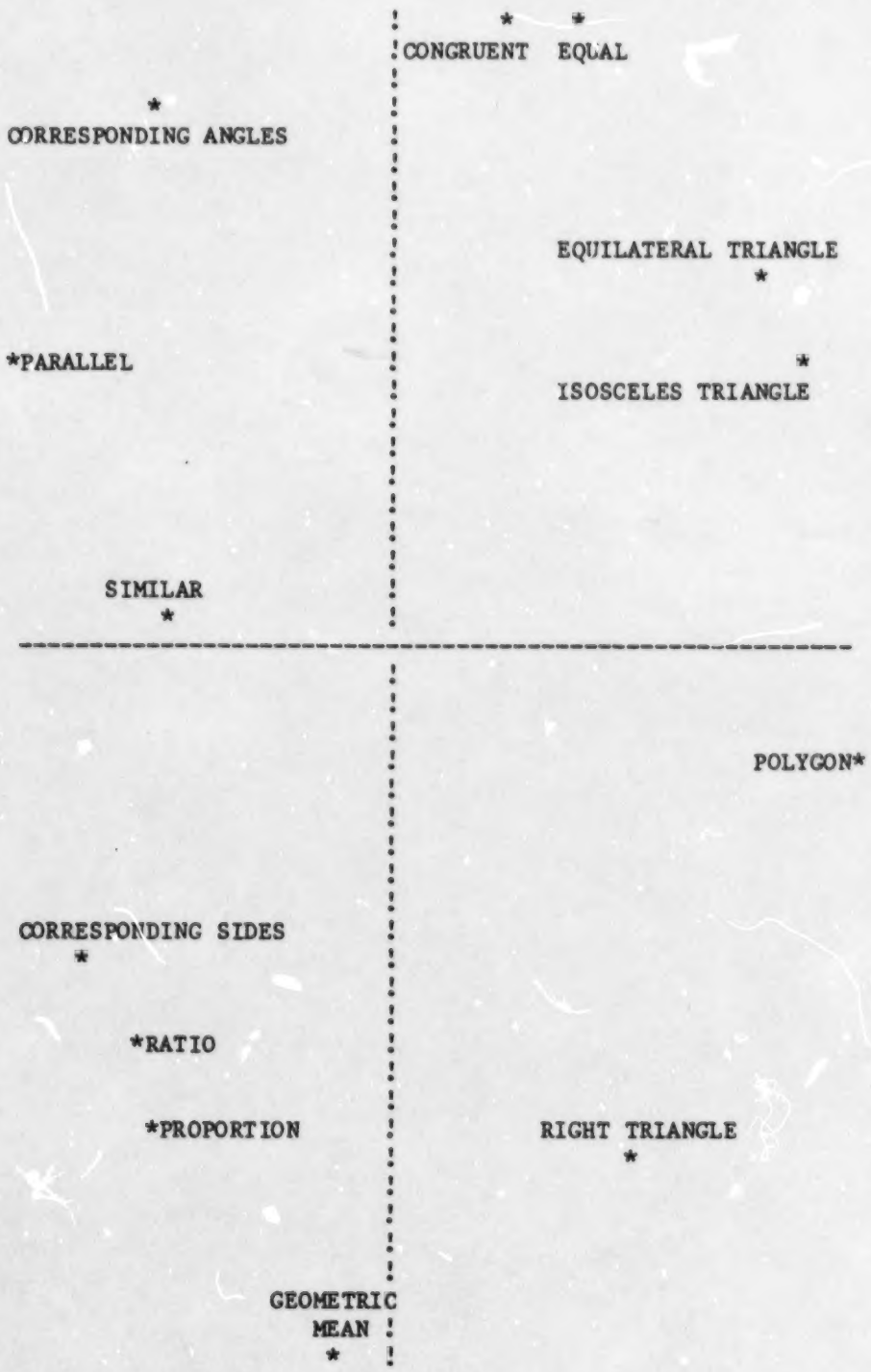

Figure 1: Expert cognitive representation - Task A

Figure 2: Prototypical concrete subject - Task A

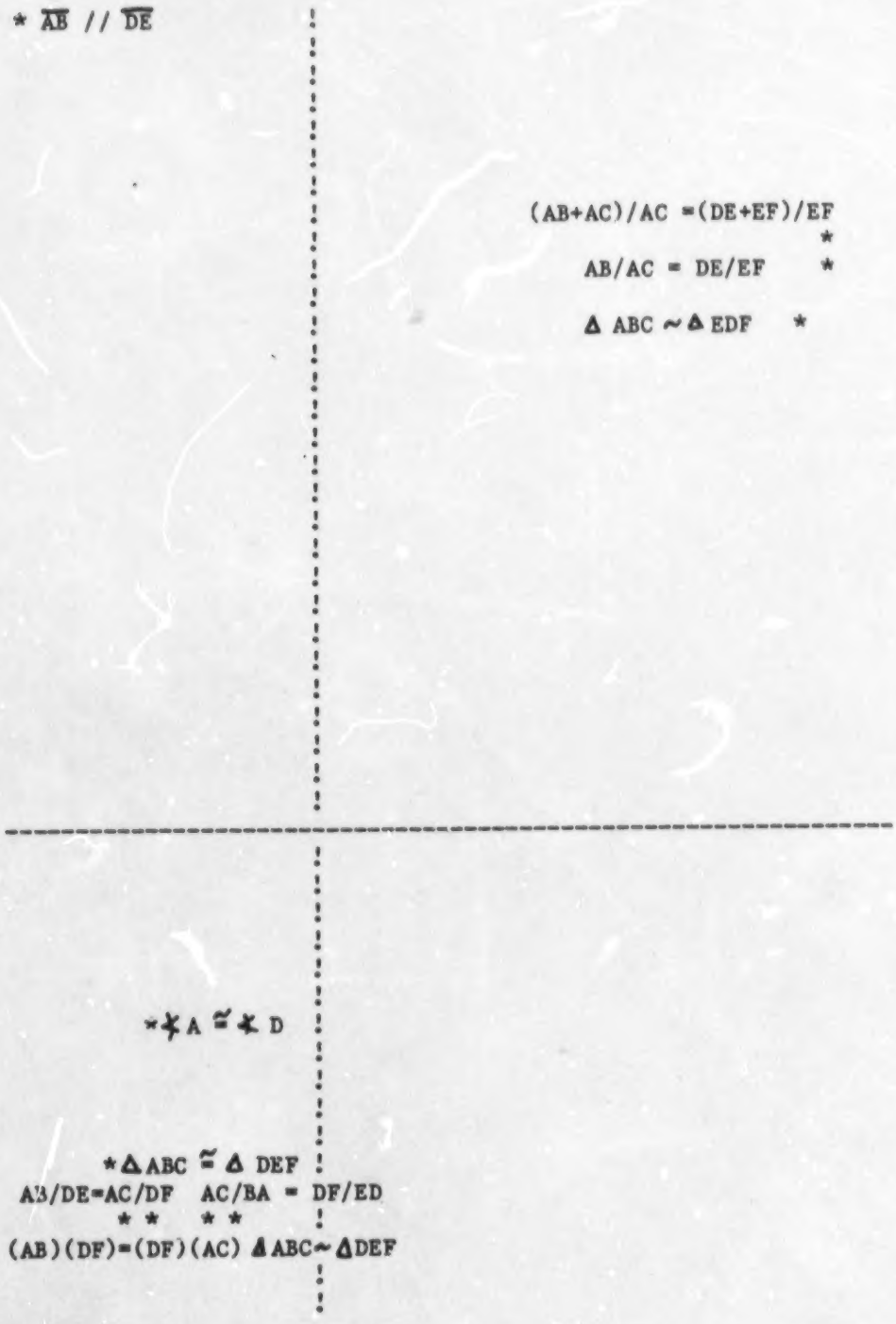

Figure 3: Expert cognitive representation - Task B

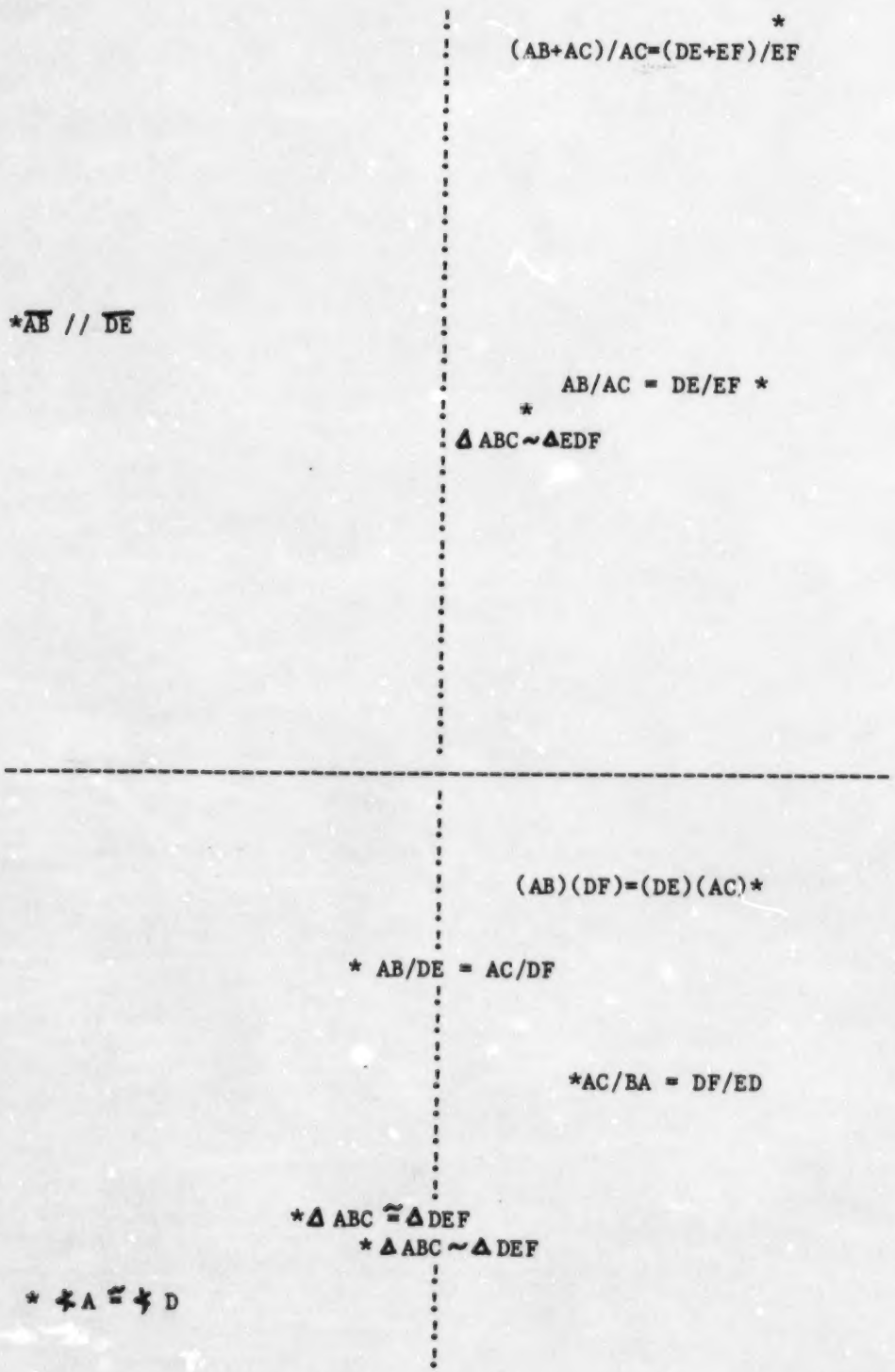

Figure 4: Prototypical concrete subject - Task B